

The Exact $SL(K+3, \mathbb{C})$ Symmetry of String Scattering Amplitudes

Sheng-Hong Lai,^{1,*} Jen-Chi Lee,^{1,†} and Yi Yang^{1,‡}

¹*Department of Electrophysics, National Chiao-Tung University, Hsinchu, Taiwan, R.O.C.*

(Dated: March 2, 2016)

We discover that the $26D$ open bosonic string scattering amplitudes (SSA) of three tachyons and one arbitrary string state can be expressed in terms of the D-type Lauricella functions with associated $SL(K+3, \mathbb{C})$ symmetry. As a result, SSA and symmetries or relations among SSA of different string states at various limits calculated previously can be rederived. These include the linear relations conjectured by Gross [1–3] and proved in [4–9] in the hard scattering limit, the recurrence relations in the Regge scattering limit [14–16] and the extended recurrence relations in the nonrelativistic scattering limit [19] discovered recently. Finally, as an application, we calculate a new recurrence relation of SSA which is valid for *all* energies.

Introduction It has long been believed that there exist huge hidden spacetime symmetries of string theory. As a consistent theory of quantum gravity, string theory contains no free parameter and an infinite number of higher spin string states. On the other hand, the very soft exponential fall-off behavior of string scattering amplitudes (SSA) in the hard scattering limit, in contrast to the power law behavior of those of quantum field theory, strongly suggests the existence of infinite number of relations among SSA of different string states. These relations or symmetries soften the UV structure of quantum string theory. Indeed, this kind of infinite relations were conjectured by Gross [1–3] and later explicitly proved in [4–9], and can be used to reduce the number of independent hard SSA from ∞ down to 1.

Historically, there were at least three approaches to probe stringy symmetries of higher spin string states. These include the gauge symmetry of Witten string field theory, the conjecture of Gross [2] on symmetries or linear relations among SSA of different string states in the hard scattering limit [1–3] and Moore’s bracket algebra approach [10–12] of stringy symmetries. See a recent review [13] for some connections of these three approaches.

Recently, it was found that the Regge SSA of three tachyons and one arbitrary string states can be expressed in terms of a sum of Kummer functions U [14–16], which soon later were shown to be the first Appell function F_1 [16]. Regge stringy symmetries or recurrence relations [15, 16] were then constructed and used to reduce the number of independent Regge SSA from ∞ down to 1. Moreover, an interesting link between Regge SSA and hard SSA was found [14, 17], and for each mass level the ratios among hard SSA can be extracted from Regge SSA. This result enables us to argue that the known $SL(5; C)$ dynamical symmetry of the Appell function F_1 [18] is crucial to probe high energy spacetime symmetry of string theory.

More recently, the extended recurrence relations [19]

among nonrelativistic low energy SSA of a class of string states with different spins and different channels were constructed by using the recurrence relations of the Gauss hypergeometry functions with associated $SL(4, \mathbb{C})$ symmetry [20]. These extended recurrence relations generalize and extend the field theory BCJ [21] relations to higher spin string states.

To further uncover the structure of stringy symmetries, in this paper we calculate the $26D$ open bosonic SSA of three tachyons and one arbitrary string states at *arbitrary* energy. We discover that these SSA can be expressed in terms of the D-type Lauricella functions with associated $SL(K+3, \mathbb{C})$ symmetry [20]. As a result, all these SSA and symmetries or relations among SSA of different string states at various limits calculated previously can be rederived. These include the linear relations conjectured by Gross [2] and proved in [4–9] in the hard scattering limit, the recurrence relations in the Regge scattering limit [15, 16] with associated $SL(5; C)$ symmetry and the extended recurrence relations in the nonrelativistic scattering limit [19] with associated $SL(4; C)$ symmetry discovered very recently.

As a byproduct from the calculation of rederiving linear relations in the hard scattering limit directly from Lauricella functions, we propose an identity which generalizes the Stirling number identity [14, 17] used previously to extract ratios among hard SSA from the Appell functions in Regge SSA. Finally, as an example, we calculate a new recurrence relation of SSA which is valid for *all* energies.

Four-point string amplitudes We will consider SSA of three tachyons and one arbitrary string states put at the second vertex. For the $26D$ open bosonic string, the general states at mass level $M_2^2 = 2(N-1)$, $N = \sum_{n,m,l \geq 0} (nr_n^T + mr_m^P + lr_l^L)$ with polarizations on the scattering plane are of the form

$$|r_n^T, r_m^P, r_l^L\rangle = \prod_{n>0} (\alpha_{-n}^T)^{r_n^T} \prod_{m>0} (\alpha_{-m}^P)^{r_m^P} \prod_{l>0} (\alpha_{-l}^L)^{r_l^L} |0, k\rangle. \quad (1)$$

*Electronic address: xgcj944137@gmail.com

†Electronic address: jcclee@cc.nctu.edu.tw

‡Electronic address: yiyang@mail.nctu.edu.tw

In the CM frame, the kinematics are defined as

$$k_1 = \left(\sqrt{M_1^2 + |\vec{k}_1|^2}, -|\vec{k}_1|, 0 \right), \quad (2)$$

$$k_2 = \left(\sqrt{M_2^2 + |\vec{k}_1|^2}, +|\vec{k}_1|, 0 \right), \quad (3)$$

$$k_3 = \left(-\sqrt{M_3^2 + |\vec{k}_3|^2}, -|\vec{k}_3| \cos \phi, -|\vec{k}_3| \sin \phi \right), \quad (4)$$

$$k_4 = \left(-\sqrt{M_4^2 + |\vec{k}_3|^2}, +|\vec{k}_3| \cos \phi, +|\vec{k}_3| \sin \phi \right) \quad (5)$$

with $M_1^2 = M_3^2 = M_4^2 = -2$ and ϕ is the scattering angle. The Mandelstam variables are $s = -(k_1 + k_2)^2$, $t = -(k_2 + k_3)^2$ and $u = -(k_1 + k_3)^2$. There are three polarizations on the scattering plane

$$e^T = (0, 0, 1), \quad (6)$$

$$e^L = \frac{1}{M_2} \left(|\vec{k}_1|, \sqrt{M_2^2 + |\vec{k}_1|^2}, 0 \right), \quad (7)$$

$$e^P = \frac{1}{M_2} \left(\sqrt{M_2^2 + |\vec{k}_1|^2}, |\vec{k}_1|, 0 \right). \quad (8)$$

For later use, we define

$$k_i^X \equiv e^X \cdot k_i \text{ for } X = (T, P, L). \quad (9)$$

Note that SSA of three tachyons and one arbitrary string state with polarizations orthogonal to the scattering plane vanish. The (s, t) and (t, u) channels SSA of states in Eq.(1) can be calculated to be

$$\begin{aligned} & A_{st}^{(r_n^T, r_m^P, r_l^L)} \\ &= B \left(-\frac{t}{2} - 1, -\frac{s}{2} - 1 \right) \prod_{n=1} [(n-1)! k_3^T]^{r_n^T} \\ & \cdot \prod_{m=1} [(m-1)! k_3^P]^{r_m^P} \prod_{l=1} [(l-1)! k_3^L]^{r_l^L} \\ & \cdot F_D^{(K)} \left(-\frac{t}{2} - 1; R_n^T, R_m^P, R_l^L; \frac{u}{2} + 2 - N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L \right), \end{aligned} \quad (10)$$

$$\begin{aligned} & A_{tu}^{(r_n^T, r_m^P, r_l^L)} \\ &= B \left(-\frac{t}{2} - 1, -\frac{u}{2} - 1 \right) \prod_{n=1} [(n-1)! k_3^T]^{r_n^T} \\ & \cdot \prod_{m=1} [(m-1)! k_3^P]^{r_m^P} \prod_{l=1} [(l-1)! k_3^L]^{r_l^L} \\ & \cdot F_D^{(K)} \left(-\frac{t}{2} - 1; R_n^T, R_m^P, R_l^L; \frac{s}{2} + 2 - N; Z_n^T, Z_m^P, Z_l^L \right), \end{aligned} \quad (11)$$

where we have defined $R_k^X \equiv \{-r_1^X\}^1, \dots, \{-r_k^X\}^k$ with $\{a\}^n = \underbrace{a, a, \dots, a}_n$, $Z_k^X \equiv [z_1^X], \dots, [z_k^X]$ with $[z_k^X] = z_{k0}^X, \dots, z_{k(k-1)}^X$ and $z_k^X = \left| \left(-\frac{k_1^X}{k_3^X} \right)^{\frac{1}{k}} \right|$, $z_{kk'}^X = z_k^X e^{\frac{2\pi i k'}{k}}$,

$\tilde{z}_{kk'}^X \equiv 1 - z_{kk'}^X$ for $k' = 0, \dots, k-1$. The integer K is defined to be

$$K = \sum_{j=1}^n j \quad + \quad \sum_{j=1}^m j \quad + \quad \sum_{j=1}^l j \quad . \quad (12)$$

{for all $r_j^T \neq 0$ } {for all $r_j^P \neq 0$ } {for all $r_j^L \neq 0$ }

For a given K , there can be SSA with different mass level N . The D-type Lauricella function $F_D^{(K)}$ is one of the four extensions of the Gauss hypergeometric function to K variables and is defined as

$$\begin{aligned} & F_D^{(K)}(a; b_1, \dots, b_K; c; x_1, \dots, x_K) \\ &= \sum_{n_1, \dots, n_K} \frac{(a)_{n_1 + \dots + n_K}}{(c)_{n_1 + \dots + n_K}} \frac{(b_1)_{n_1} \dots (b_K)_{n_K}}{n_1! \dots n_K!} x_1^{n_1} \dots x_K^{n_K} \end{aligned} \quad (13)$$

where $(a)_n = a(a+1) \dots (a+n-1)$ is the Pochhammer symbol. There was a integral representation of the Lauricella function $F_D^{(K)}$ discovered by Appell and Kampe de Fériet (1926) [22]

$$\begin{aligned} & F_D^{(K)}(a; b_1, \dots, b_K; c; x_1, \dots, x_K) \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt t^{a-1} (1-t)^{c-a-1} \\ & \cdot (1-x_1 t)^{-b_1} (1-x_2 t)^{-b_2} \dots (1-x_K t)^{-b_K}, \end{aligned} \quad (14)$$

which was used to calculate Eq.(10) and Eq.(11). By using the identity of Lauricella function for $b_i \in \mathbb{Z}^-$

$$\begin{aligned} & F_D^{(K)}(a; b_1, \dots, b_K; c; x_1, \dots, x_K) = \frac{\Gamma(c) \Gamma(c-a-\sum b_i)}{\Gamma(c-a) \Gamma(c-\sum b_i)} \\ & \cdot F_D^{(K)} \left(a; b_1, \dots, b_K; 1+a+\sum b_i - c; 1-x_1, \dots, 1-x_K \right), \end{aligned} \quad (15)$$

we can rederive the string BCJ relation [19]

$$\begin{aligned} & \frac{A_{st}^{(r_n^T, r_m^P, r_l^L)}}{A_{tu}^{(r_n^T, r_m^P, r_l^L)}} = \frac{\Gamma(-\frac{s}{2}-1) \Gamma(\frac{s}{2}+2)}{\Gamma(\frac{u}{2}+2-N) \Gamma(-\frac{u}{2}-1+N)} \\ &= \frac{\sin(\frac{\pi u}{2})}{\sin(\frac{\pi s}{2})} = \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 \cdot k_2)}, \end{aligned} \quad (16)$$

which gives another form of the (s, t) channel amplitude

$$\begin{aligned} & A_{st}^{(r_n^T, r_m^P, r_l^L)} \\ &= B \left(-\frac{t}{2} - 1, -\frac{s}{2} - 1 + N \right) \prod_{n=1} [(n-1)! k_3^T]^{r_n^T} \\ & \cdot \prod_{m=1} [(m-1)! k_3^P]^{r_m^P} \prod_{l=1} [(l-1)! k_3^L]^{r_l^L} \\ & \cdot F_D^{(K)} \left(-\frac{t}{2} - 1; R_n^T, R_m^P, R_l^L; \frac{s}{2} + 2 - N; Z_n^T, Z_m^P, Z_l^L \right). \end{aligned} \quad (17)$$

Regge scattering limit The relevant kinematics in Regge limit are

$$k_1^T = 0, \quad k_3^T \simeq -\sqrt{-t}, \quad (18)$$

$$k_1^P \simeq -\frac{s}{2M_2}, \quad k_3^P \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}, \quad (19)$$

$$k_1^L \simeq -\frac{s}{2M_2}, \quad k_3^L \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2}, \quad (20)$$

with $\tilde{z}_{kk'}^T = 1$, $\tilde{z}_{kk'}^P = 1 - \left(-\frac{s}{t}\right)^{1/k} e^{\frac{i2\pi k'}{k}} \sim s^{1/k}$ and $\tilde{z}_{kk'}^L = 1 - \left(-\frac{s}{t'}\right)^{1/k} e^{\frac{i2\pi k'}{k}} \sim s^{1/k}$. In the Regge limit, the SSA in Eq.(10) reduces to

$$\begin{aligned} & A_{st}^{(r_n^T, r_m^P, r_l^L)} \\ & \simeq B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) \prod_{n=1} [(n-1)! \sqrt{-t}]^{r_n^T} \\ & \cdot \prod_{m=1} \left[(m-1)! \frac{\tilde{t}}{2M_2}\right]^{r_m^P} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2}\right]^{r_l^L} \\ & \cdot F_1\left(-\frac{t}{2} - 1; -q_1, -r_1; -\frac{s}{2}; \frac{s}{t}, \frac{s}{t'}\right), \end{aligned} \quad (21)$$

where F_1 is the Appell function. Eq.(21) agrees with the result obtained in [16] previously.

Hard scattering limit In the hard scattering limit $e^P = e^L$ [4, 5], we can consider only the polarization e^L case. The relevant kinematics are

$$k_1^T = 0, \quad k_3^T \simeq -E \sin \phi, \quad (22)$$

$$k_1^L \simeq -\frac{2p^2}{M_2} \simeq -\frac{2E^2}{M_2}, \quad (23)$$

$$k_3^L \simeq \frac{2E^2}{M_2} \sin^2 \frac{\phi}{2}, \quad (24)$$

with $\tilde{z}_{kk'}^T = 1$, $\tilde{z}_{kk'}^L = 1 - \left(-\frac{s}{t}\right)^{1/k} e^{\frac{i2\pi k'}{k}} \sim O(1)$. The SSA in Eq.(10) reduces to

$$\begin{aligned} & A_{st}^{(r_n^T, r_l^L)} = B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) \\ & \cdot \prod_{n=1} [(n-1)! E \sin \phi]^{r_n^T} \prod_{l=1} \left[-(l-1)! \frac{2E^2}{M_2} \sin^2 \frac{\phi}{2}\right]^{r_l^L} \\ & \cdot F_D^{(K)}\left(-\frac{t}{2} - 1; R_n^T, R_l^L; \frac{u}{2} + 2 - N; (1)_n, \tilde{Z}_l^L\right). \end{aligned} \quad (25)$$

One key observation of the previous hard SSA calculation [4–9] was that there was a difference between the naive energy order and the real energy order corresponding to the $(\alpha_{-1}^L)^{r_1^L}$ operator in Eq.(1). So let's pay attention to

the corresponding summation and write

$$\begin{aligned} & A_{st}^{(r_n^T, r_l^L)} = B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) \\ & \cdot \prod_{n=1} [(n-1)! E \sin \phi]^{r_n^T} \prod_{l=1} \left[-(l-1)! \frac{2E^2}{M_2} \sin^2 \frac{\phi}{2}\right]^{r_l^L} \\ & \cdot \sum_{k_r} \frac{\left(-\frac{t}{2} - 1\right)_{k_r}}{\left(\frac{u}{2} + 2 - N\right)_{k_r}} \frac{(-r_1^L)_{k_r}}{k_r!} \left(1 + \frac{s}{t}\right)^{k_r} \cdot (\dots) \end{aligned} \quad (26)$$

where we have used $(a)_{n+m} = (a)_n (a+n)_m$. We then propose the following formula

$$\begin{aligned} & \sum_{k_r=0}^{r_1} \frac{\left(-\frac{t}{2} - 1\right)_{k_r}}{\left(\frac{u}{2} + 2 - N\right)_{k_r}} \frac{(-r_1^L)_{k_r}}{k_r!} \left(1 + \frac{s}{t}\right)^{k_r} \\ & = 0 \cdot \left(\frac{tu}{s}\right)^0 + 0 \cdot \left(\frac{tu}{s}\right)^{-1} + \dots + 0 \cdot \left(\frac{tu}{s}\right)^{-\left[\frac{r_1^L+1}{2}\right]-1} \\ & + C_{r_1^L} \left(\frac{tu}{s}\right)^{-\left[\frac{r_1^L+1}{2}\right]} + O\left\{\left(\frac{tu}{s}\right)^{-\left[\frac{r_1^L+1}{2}\right]+1}\right\}, \end{aligned} \quad (27)$$

which is a generalization of the Stirling number identity proposed in [14] and proved in [17]. In Eq.(27), $C_{r_1^L}$ is independent of energy E and depends on r_1^L and possibly scattering angle ϕ , and the 0 terms correspond to the naive energy order in the hard SSA calculation. The leading order SSA in the hard scattering limit can then be identified

$$\begin{aligned} & A_{st}^{(r_n^T, r_l^L)} \simeq B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) \\ & \cdot \prod_{n=1} [(n-1)! E \sin \phi]^{r_n^T} \prod_{l=1} \left[-(l-1)! \frac{2E^2}{M_2} \sin^2 \frac{\phi}{2}\right]^{r_l^L} \\ & \cdot C_{r_l^L} (E \sin \phi)^{-2\left[\frac{r_l^L+1}{2}\right]} \cdot (\dots) \\ & \sim E^{N - \sum_{n \geq 2} n r_n^T - \left(2\left[\frac{r_1^L+1}{2}\right] - r_1^L\right) - \sum_{l \geq 3} l r_l^L} \\ & \Rightarrow r_{n \geq 2}^T = r_{l \geq 3}^L = 0 \text{ and } r_1^L = 2m, \end{aligned} \quad (28)$$

which means for $r_l^L = 1, 3, 5, \dots$, the amplitudes are of subleading order in energy. This is consistent with the previous results [4–9]. We further propose that $C_{r_l^L} = \frac{(2m)!}{m!}$ and is ϕ independent for $r_l^L = 2m$ in Eq.(27). We have verified Eq.(27) for $r_1 = 0, 1, 2, \dots, 10$. Finally the

leading order SSA in the hard scattering limit, i.e. $r_1^T = N - 2m - 2$, $r_1^L = 2m$ and $r_2^L = q$, can be calculated to be

$$\begin{aligned} & A_{st}^{(N-2m-2q, 2m, q)} \\ & \simeq B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) (E \sin \phi)^N \frac{(2m)!}{m!} \left(-\frac{1}{2M_2}\right)^{2m+q} \\ & = (2m-1)!! \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} A_{st}^{(N, 0, 0)} \end{aligned} \quad (29)$$

which is consistent with the previous result [4–9].

Nonrelativistic scattering limit In this limit $|\vec{k}_1| \ll M_2$, we have

$$k_1^T = 0, k_3^T = -\left[\frac{\epsilon}{2} + \frac{(M_1 + M_2)^2}{4M_1M_2\epsilon}|\vec{k}_1|^2\right] \sin \phi, \quad (30)$$

$$k_1^L = -\frac{M_1 + M_2}{M_2}|\vec{k}_1| + O(|\vec{k}_1|^2), \quad (31)$$

$$k_3^L = -\frac{\epsilon}{2} \cos \phi + \frac{M_1 + M_2}{2M_2}|\vec{k}_1| + O(|\vec{k}_1|^2), \quad (32)$$

$$k_1^P = -M_1 + O(|\vec{k}_1|^2), \quad (33)$$

$$k_3^P = \frac{M_1 + M_2}{2} - \frac{\epsilon}{2M_2} \cos \phi |\vec{k}_1| + O(|\vec{k}_1|^2) \quad (34)$$

where $\epsilon = \sqrt{(M_1 + M_2)^2 - 4M_3^2}$, and $z_k^T = z_k^L = 0$, $z_k^P \simeq \left|\left(\frac{2M_1}{M_1 + M_2}\right)^{\frac{1}{k}}\right|$. The SSA in Eq.(17) reduces to

$$\begin{aligned} & A_{st}^{(r_n^T, r_m^P, r_l^L)} \\ & \simeq \prod_{n=1} \left[(n-1)! \frac{\epsilon}{2} \sin \phi\right]^{r_n^T} \prod_{m=1} \left[-(m-1)! \frac{M_1 + M_2}{2}\right]^{r_m^P} \\ & \cdot \prod_{l=1} \left[(l-1)! \frac{\epsilon}{2} \cos \phi\right]^{r_l^L} B\left(\frac{M_1 M_2}{2}, 1 - M_1 M_2\right) \\ & \cdot F_D^{(K)}\left(\frac{M_1 M_2}{2}; R_m^P; M_1 M_2; \left(\frac{2M_1}{M_1 + M_2}\right)_m\right), \end{aligned} \quad (35)$$

where $K = \sum_{j=1}^m j$. Note that for string states with $r_k^P = 0$ for all $k \geq 2$, one has $K = 1$ and the Lauricella functions in the low energy nonrelativistic SSA reduce to the Gauss hypergeometry functions $F_D^{(1)} = {}_2F_1$. In particular, for the case of $r_1^T = N_1$, $r_1^P = N_3$, $r_1^L = N_2$, and $r_k^X = 0$ for all $k \geq 2$, the SSA reduces to

$$\begin{aligned} & A_{st}^{(N_1, N_2, N_3)} \\ & = \left(\frac{\epsilon}{2} \sin \phi\right)^{N_1} \left(\frac{\epsilon}{2} \cos \phi\right)^{N_2} \\ & \cdot \left(-\frac{M_1 + M_2}{2}\right)^{N_3} B\left(\frac{M_1 M_2}{2}, 1 - M_1 M_2\right) \\ & \cdot {}_2F_1\left(\frac{M_1 M_2}{2}; -N_3; M_1 M_2; \frac{2M_1}{M_1 + M_2}\right), \end{aligned} \quad (36)$$

which agrees with the result obtained in [19] previously. *Exact symmetry of string scattering amplitudes* In the Lie group approach of special functions, the associated Lie group for the Lauricella function $F_D^{(K)}$ in the SSA at each fixed K is the $SL(K+3, \mathbb{C})$ group [20] which contains the $SL(2, \mathbb{C})$ fundamental representation of the $3+1$ dimensional spacetime Lorentz group $SO(3, 1)$. So $sl(K+3, \mathbb{C})$ contains the $2+1$ dimensional $so(2, 1)$, the Lorentz spacetime symmetry in our case as well. In the Regge limit, the Lauricella function in the SSA reduces to the Appell function F_1 with associated group $SL(5, \mathbb{C})$ [18], which is K independent. In the low energy nonrelativistic limit, the Lauricella function in the SSA reduces to the Gauss hypergeometry function ${}_2F_1$ with associated group $SL(4, \mathbb{C})$ [20], which is also K independent.

In sum, we have identified the *exact* $SL(K+3, \mathbb{C})$ symmetry of string scattering amplitudes with three tachyons and one *arbitrary* string state of $26D$ bosonic open string theory. Finally, with the $SL(K+3, \mathbb{C})$ group and the recurrence relations of the Lauricella functions $F_D^{(K)}$, one can derive infinite number of recurrence relations of SSA of different string states which are valid for *all* energies. For a simple example, the following recurrence relation of $F_D^{(K)}$ can be verified

$$\begin{aligned} & cF_D^{(K)}(b_j; c) + c(x_j - 1)F_D^{(K)}(b_j + 1; c) \\ & + (a - c)x_j F_D^{(K)}(b_j + 1; c + 1) = 0, \end{aligned} \quad (37)$$

which leads to the recurrence relation of SSA

$$\left(\frac{u}{2} + 2 - N\right) A_{st}^{(r_n^T, r_m^P, r_l^L)} - \left(\frac{s}{2} + 1\right) k_3^T A_{st}^{(r_n^T, r_m^P, r_l^L)} = 0 \quad (38)$$

where (r_n^T, r_m^P, r_l^L) means the group $(-\{r_1^T - 1\}^1, \{-r_2^T\}^2, \dots, \{-r_n^T\}^n; R_m^P, R_l^L)$ of polarizations. In Eq.(37), we have omitted those arguments of $F_D^{(K)}$ which remain the same for all three Lauricella functions:

Acknowledgments

This work is supported in part by the Ministry of Science and Technology and S.T. Yau center of NCTU, Taiwan.

-
- [1] D. J. Gross and P. F. Mende, Phys. Lett. B **197**, 129 (1987); Nucl. Phys. B **303**, 407 (1988).
 - [2] D. J. Gross, Phys. Rev. Lett. **60**, 1229 (1988); D. J. Gross and J. R. Ellis, Phil. Trans. R. Soc. Lond. A329, 401 (1989).
 - [3] D. J. Gross and J. L. Manes, Nucl. Phys. B **326**, 73 (1989). See section 6 for details.
 - [4] C. T. Chan and J. C. Lee, Phys. Lett. B **611**, 193 (2005).

- J. C. Lee, [arXiv:hep-th/0303012].
- [5] C. T. Chan and J. C. Lee, Nucl. Phys. B **690**, 3 (2004).
- [6] C. T. Chan, P. M. Ho and J. C. Lee, Nucl. Phys. B **708**, 99 (2005).
- [7] C. T. Chan, P. M. Ho, J. C. Lee, S. Teraguchi and Y. Yang, Phys. Rev. Lett. **96** (2006) 171601, hep-th/0505035.
- [8] C. T. Chan, P. M. Ho, J. C. Lee, S. Teraguchi and

- Y. Yang, Nucl. Phys. B **725**, 352 (2005).
- [9] C. T. Chan, J. C. Lee and Y. Yang, Nucl. Phys. B **738**, 93 (2006).
 - [10] Gregory Moore, Finite in all directions. arXiv:hep-th/9305139, 1993.
 - [11] Gregory Moore, Symmetries of the bosonic string S-matrix. arXiv:hep-th/9310026, 1993.
 - [12] C.T. Chan, S. Kawamoto and D. Tomino, Nucl. Phys. B **885**, 225 (2014).
 - [13] J.C. Lee and Y. Yang, Review on High energy String Scattering Amplitudes and Symmetries of String Theory, arXiv: 1510.03297.
 - [14] S.L. Ko, J.C. Lee and Y. Yang, JHEP, 9060:028 (2009).
 - [15] J.C. Lee and Y. Mitsuka, JHEP 1304:082 (2013).
 - [16] J.C. Lee and Y. Yang, Phys. Lett. B **739**, 370 (2014).
 - [17] J.C. Lee, C. H. Yan, and Y. Yang, "High energy string scattering amplitudes and signless Stirling number identity", SIGMA, 8:045, (2012).
 - [18] Willard Miller. Jr., "Lie theory and the Appell functions F_1 ", SIAM J. Math. Anal. Vol. 4 No. 4, 638 (1973).
 - [19] S.H. Lai, J.C. Lee and Y. Yang, arXiv: 1601.0381.
 - [20] Willard Miller. Jr., "Lie theory and generalizations of the hypergeometric functions", SIAM J. Appl. Math. Vol. 25 No. 2, 226 (1973).
 - [21] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D **78**, 085011 (2008) [hep-ph/0805.3993].
 - [22] Joseph Kampe de Fariet and Paul Appell. Fonctions hypergeometriques et hyperspheriques 1926.